Algebraic Cartan Elimination in Minimal Einstein–Cartan–Holst Gravity: Spin-Sourced Contact and Zero-Spin Scalar Branches

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(Dated: July 14, 2026, 1:59 PDT (v1A.0.123))

We consolidate two standard consequences of the same algebraic Cartan equation in minimal Einstein–Cartan–Holst (ECH) gravity. On the spin-sourced branch, eliminating the non-propagating connection gives the minimal axial–axial contact interaction $-(3\kappa/16)[\gamma^2/(1+\gamma^2)]J_5^2$. A deliberately elevated homogeneous normalization illustrates only its scale: $\kappa n_\psi^2/\rho_\Lambda \simeq 3.6 \times 10^{-69}(n_\psi/100\,\mathrm{cm}^{-3})^2$. This coefficient-one dimensional benchmark omits the actual contact factor $3/16$ and the finite-Holst factor $\gamma^2/(1+\gamma^2)$; number density also does not fix the state-dependent renormalized composite $\langle J_5^I J_{5I}\rangle$, a vacuum stress tensor, or an equation of state. In the declared direct-channel, hard-four-momentum-cutoff, standard mean-field NJL convention, the scalar Fierz projection is repulsive, $G_s = -3\kappa/16$, so its real homogeneous scalar gap equation has no nonzero solution. This conditional sign result does not exclude other truncations, species structures, non-minimal couplings, or propagating torsion.

On the zero-spin branch, canonical scalar matter has no Lorentz-connection source; for an invertible tetrad and real nonsingular constant γ, the same algebraic equation gives $C = T = 0$. After solving that equation, the local classical reduced action is the Einstein–scalar action because the Holst contraction vanishes pointwise by the algebraic Bianchi identity. Thus the classical scalar equations and tensor evolution operators equal their GR counterparts for matched background and initial data and boundary data with standard falloff, so the first-order variational surface contribution vanishes. Equality of right- and left-helicity solutions additionally requires matched parity-symmetric initial data. This on-connection-shell statement is not an off-shell equality of the original first-order actions and excludes quantum/anomaly, non-minimal, propagating-torsion, and nontrivial global/topological sectors.

The identities used here are standard. The contribution is their convention-audited consolidation into the two Cartan branches and the sharply bounded dimensional coefficient benchmark above; no ECH dark-energy or birefringence prediction is made.

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I. INTRODUCTION

In Einstein–Cartan gravity the Lorentz connection is independent of the tetrad, and fermionic spin sources torsion. For the minimal action the connection equation is algebraic: torsion does not propagate and may be eliminated exactly, leaving a local four-fermion interaction [1–3]. The Holst extension introduces the dimensionless Barbero–Immirzi parameter but no new propagating degree of freedom when that parameter is constant. These elementary facts permit two useful, auditable questions: how large can the induced contact channel be at late times, and can the constant-Immirzi Holst term affect classical perturbations sourced only by canonical scalars?

This Note answers those two questions within an intentionally narrow domain. The underlying Cartan elimination, axial contact interaction, and torsion-free Bianchi identity are standard. Our contribution is to place their normalizations, operator ordering, logical branches, and observable scope in one convention-audited derivation.

Section II states the first-order action and the sourced connection equation. Section III A gives a finite-density scale illustration and a conditional standard mean-field NJL check. Appendix A records the Fierz convention and Appendix B the regulated gap equation. Section IV gives the local classical zero-spin-branch argument.

The claim boundary is as important as the calculation. A Fierz identity for the contact interaction does not enumerate the gravitational EFT. A mean-field NJL calculation does not eliminate Fierz ambiguity or beyond-mean-field strong-coupling completions. Likewise, vanishing of the Holst density on the torsion-free scalar branch is not a theorem about fermionic, quantum, non-minimal, dynamical-Immirzi, or propagating-torsion sectors. Earlier exploratory mappings from one-loop Holst/Immirzi running to late-time dark energy are not retained as results: the cited calculations do not derive the required cosmological stress tensor and observable matching [4, 5]. No operator-complete or unrestricted no-go is claimed.

II. MINIMAL ECH AND ALGEBRAIC TORSION

We use the first-order Einstein–Cartan–Holst action with a constant Barbero–Immirzi parameter γ and a minimally coupled Dirac field,

$$S[e,\omega,\psi] = \frac{1}{4\kappa}\int\left(\epsilon_{IJKL} + \frac{2}{\gamma}\eta_{I[K}\eta_{L]J}\right) \times e^I\wedge e^J\wedge R^{KL}(\omega) + S_{\rm D}[e,\omega,\psi], \tag{1}$$

where $\kappa = 8\pi G = 8\pi/M_{\rm Pl}^2 = 1/\bar{M}_{\rm Pl}^2$, $M_{\rm Pl}\equiv G^{-1/2} = 1.22089\times10^{19}$ GeV is the unreduced Planck mass, $\bar{M}_{\rm Pl}\equiv(8\pi G)^{-1/2}$ is the reduced Planck mass, $R^{IJ}(\omega) = d\omega^{IJ} + \omega^I{}_K\wedge\omega^{KJ}$, and e^I and ω^{IJ} are varied independently. There is no independent T^2 term in Eq. (1); any torsion-square or four-fermion expression below is an on-shell result obtained only after the connection equation has been solved. This ordering avoids double counting.

All sign-sensitive statements use the following single convention set. The Lorentzian internal metric is mostly plus, $\eta_{IJ} = {\rm diag}(-,+,+,+)$; $\epsilon_{0123} = +1$ (hence $\epsilon^{0123} = -1$); $\{\gamma^I,\gamma^J\} = 2\eta^{IJ}$; and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Antisymmetrization carries unit weight. The torsion two-form and full-weight component convention are $T^I = de^I+\omega^I{}_J\wedge e^J = \frac{1}{2}T^I{}_{\mu\nu}dx^\mu\wedge dx^\nu$ and $T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}$. Coincident bilinear products below are renormalized composite operators with their displayed field order. Appendix A exchanges the middle Grassmann fields to map the ordering (12)(34) to (14)(32); its one exchange supplies the minus sign relative to the Nieves–Pal c-number identity. The resulting scalar row is inserted into the explicitly declared interaction $+G_s(\bar\psi\psi)^2$, for which $G_s<0$ is repulsive in the gap equation of Appendix B.

The Holst-modified Cartan operator is nonsingular on the paper's real constant-γ domain. Define the internal dual on Lorentz bivectors by $(\star X)_{IJ} = \frac{1}{2}\epsilon_{IJ}{}^{KL}X_{KL}$, so $\star^2 = -\mathbb{1}$. Acting on an antisymmetric bivector, the tensor in parentheses in Eq. (1) is twice

$$\mathcal{Q}_\gamma = \star + \gamma^{-1}\mathbb{1}, \qquad \mathcal{Q}_\gamma^{-1} = \frac{\gamma^2}{1+\gamma^2}\left(\gamma^{-1}\mathbb{1} - \star\right), \tag{2}$$

as is verified by direct multiplication. This is the same $\gamma^2/(1+\gamma^2)$ bivector inverse displayed, in the dualized convention, by Freidel, Minic, and Takeuchi [3], Eq. (6). It exists for every real finite nonzero γ; $\gamma=0$ is outside the action written in Eq. (1), while the complex self-dual values $\gamma=\pm i$ make $1+\gamma^2=0$ and are explicitly excluded. Therefore, writing the Lorentz-connection source three-form as $\mathcal{J}^{IJ}\propto\delta S_{\rm matter}/\delta\omega_{IJ}$, the connection variation has the explicit sourced form

$$\frac{\delta S}{\delta\omega_{IJ}} = 0: \qquad \mathcal{Q}_\gamma\Big(e^{[I}\wedge T^{J]}\Big) = \mathcal{J}^{IJ}. \tag{3}$$

The normalization of $\mathcal{J}^{IJ}$ is fixed by this equation and will not be needed below. A canonical scalar action is independent of ω_{IJ}, so $\mathcal{J}^{IJ}=0$ and invertibility of $\mathcal{Q}_\gamma$ gives $e^{[I}\wedge T^{J]}=0$. For completeness, let E_I be the frame dual to e^I and define the one-form $t\equiv\iota_{E_I}T^I$ (summed over I). Contracting the last equation with ι_{E_I} in four dimensions gives $T^J + e^J\wedge t = 0$. A second contraction yields $t = \iota_{E_J}T^J = -3t$, hence $t=0$ and therefore $T^I=0$. Thus the kernel is trivial for an invertible tetrad. The Einstein–Cartan limit $\gamma\to\infty$ is nonsingular as well.

For minimal fermion coupling the spin current is totally antisymmetric and dual to $J_5^I = \bar\psi\gamma^I\gamma^5\psi$. The connection equation is algebraic. In the Einstein–Cartan limit, with the spin-current convention $S^{IJK} = \frac{1}{4}\epsilon^{IJKL}J_{5L}$ used here, it reads

$$T^{IJK} = \kappa S^{IJK} = \frac{\kappa}{4}\epsilon^{IJKL}J_{5L}. \tag{4}$$

At finite γ, Freidel, Minic, and Takeuchi vary the connection explicitly and obtain, for minimal coupling and their axial current $A^I = J_5^I$,

$$e_I{}^\mu C_{\mu JK} = 4\pi G\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{2}\epsilon_{IJKL}J_5^L - \frac{1}{\gamma}\eta_{I[J}J^5_{K]}\right), \tag{5}$$

their Eq. (17) [3]. This source-backed solution makes the zero-current implication immediate: $J_5^I=0$ gives $C_{\mu JK}=0$ and hence $T^I=0$. Their direct back-substitution, Eq. (23), gives $\mathcal{L}_{\rm int} = -(3/2)\pi G[\gamma^2/(1+\gamma^2)]J_5^2$ for minimal coupling. In the convention of Eqs. (3) and (5), the source-to-contact normalization bridge is

$$4\pi G = \frac{\kappa}{2}, \qquad -\frac{3}{2}\pi G = -\frac{3\kappa}{16}, \tag{6}$$

so the source solution and back-substituted coefficient are in the same Freidel–Minic–Takeuchi normalization [3]. It gives

$$\mathcal{L}_{4\psi} = -\frac{3\kappa}{16}\frac{\gamma^2}{1+\gamma^2}\left(J_5^I J_{5I}\right). \tag{7}$$

The pure Einstein–Cartan coefficient is recovered for $\gamma \to \infty$. A vector–axial term is not generated by the minimal coupling used here; it requires a non-minimal fermion coupling and is outside this paper's declared field content.

For the late-time dimensional benchmark and gap-equation calculation below we use the maximal Einstein–Cartan magnitude,

$$\mathcal{L}_{\rm tor}^{\rm NJL} = -\frac{3\kappa}{16}(J_5^I J_{5I}), \tag{8}$$

because $\gamma^2/(1+\gamma^2) < 1$ can only reduce the coupling. Equations (1)–(8) are the complete action-level input to the results that follow.

III. THE MINIMAL FOUR-FERMION CONTACT CHANNEL

The contact operator in Eq. (9) is a specific consequence of minimal algebraic torsion. We test its finite-density amplitude and its standard mean-field scalar-condensate channel. Neither test is promoted to a claim about every four-fermion completion or about the complete gravitational EFT.

A. Finite-density benchmark and equation-of-state boundary

On the standard Einstein–Cartan side—i.e. before the Holst term is added—the Cartan algebraic equation $T^{abc} = \kappa\, S^{abc}$ (Eq. (4); in the half-weight torsion convention $T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]}$ of the original literature this reads $T^{abc} = (\kappa/2)\,\bar\psi\gamma^{[a}\gamma^b\gamma^{c]}\psi$ — the two map exactly, because the half-weight definition absorbs the factor of two relative to the full-weight definition used in Eq. (4)) allows torsion to be integrated out exactly, generating an effective four-fermion contact term whose coefficient is the gravitational coupling $\kappa = 8\pi G$ [1, 2]. Following the standard Hehl–Datta derivation, the resulting axial–axial contact interaction is

$$\mathcal{L}_{\rm tor}^{\rm NJL} = -\,\frac{3}{16}\,\kappa\,\big(\bar\psi\,\gamma^a\gamma^5\,\psi\big)^2, \tag{9}$$

i.e. a four-fermion operator suppressed by $M_{\rm Pl}^{-2}$ and *parity-even* in the *CP*-conserving Standard Model sector. For a specified homogeneous number density we define the coefficient-one dimensional benchmark $\rho_{4f}^{(\rm dim)} \equiv \kappa n_\psi^2 = 8\pi n_\psi^2/M_{\rm Pl}^2$. The actual maximal axial–axial coefficient in Eq. (9) adds the factor 3/16. Since n_ψ has mass dimension +3, both expressions have the required energy-density dimension +4. This definition is not an inequality for $|\langle J_5^I J_{5I}\rangle|$: number density does not determine that state-dependent coincident composite without specifying the ensemble, spin polarization, relativistic normalization, species contractions, and a renormalized contact prescription. Using $\hbar c = 1.9733 \times 10^{-5}\,{\rm eV\,cm}$ and $M_{\rm Pl} = 1.2209 \times 10^{28}\,{\rm eV}$ gives the transparent scaling

$$\kappa n_\psi^2 \simeq 1.0 \times 10^{-79}\left(\frac{n_\psi}{100\,{\rm cm}^{-3}}\right)^2 {\rm eV}^4,$$
$$\frac{\kappa n_\psi^2}{\rho_\Lambda} \simeq 3.6 \times 10^{-69}\left(\frac{n_\psi}{100\,{\rm cm}^{-3}}\right)^2. \tag{10}$$

The normalization $100\,{\rm cm}^{-3}$ is deliberately elevated for illustration; it is neither a cosmological-density estimate nor a preferred state. At that normalization the coefficient-one benchmark is about 68.4 orders below the canonical $\rho_\Lambda \approx (2.3\,{\rm meV})^4 \approx 2.8 \times 10^{-11}\,{\rm eV}^4$ used throughout this paper. Including the actual 3/16 contact coefficient gives the coefficient-weighted benchmark $1.9 \times 10^{-80}\,{\rm eV}^4 = 6.7 \times 10^{-70}\rho_\Lambda$. No coherent order parameter, vacuum expectation, stress tensor, or equation of state is inferred from $\langle J^5\rangle \approx 0$: a one-point current does not determine the state-dependent composite $\langle J^5 J^5\rangle$ or its metric variation. The only late-density statement made here is the explicit finite-density coefficient-scale comparison above. This numerical comparison alone is not a constraint on the actual composite expectation or on arbitrary torsion and fermion sectors. It illustrates the fermionic spin-sourced branch and must not be conflated with the canonical scalar, zero-spin theorem of Sec. IV. The finite-γ coefficient $\gamma^2/(1+\gamma^2)$ in Eq. (7) has magnitude below unity for real finite γ, so it does not increase this coefficient benchmark above the Einstein–Cartan limit used in the calculation.

B. Standard mean-field NJL check

The finite-density benchmark does not decide whether a vacuum condensate exists. As a logically separate conditional check, we Fierz-project the axial–axial operator using the exchange ordering and Grassmann sign fixed in Appendix A, and solve a hard-cutoff NJL gap equation in Appendix B. In the declared direct-channel convention the scalar projection is $G_{\rm scalar} = -3\kappa/16 < 0$, hence repulsive. Substitution in the real homogeneous scalar gap equation shows that this negative coupling has no nonzero solution. The sign conclusion is logically separate from the coefficient-magnitude diagnostic: at the retained EFT bookkeeping ceiling $\Lambda = M_{\rm Pl}$, the three displayed scalar ratios run from 0.239 to 2.15 and therefore are not uniformly subcritical. We do not evaluate the contact EFT above $M_{\rm Pl}$. The magnitude $|G_A| = 3\kappa/32$ divided by the same scalar-channel threshold is one half of the scalar ratio; it is not an independently derived axial-condensation threshold. The three-row scan is reported in Appendix B. The machine-checkable calculation is `njl_gap_equation_route1.py`.

This result is deliberately conditional. It closes the scalar condensate in the standard mean-field NJL model with the stated regulator, field content, and projection convention. It does not remove the known Fierz ambiguity of mean-field truncations and does not exclude beyond-mean-field strong coupling, additional flavor/color exchange structure, or non-minimal torsion couplings.

a. Running literature. The one-loop external-current calculation of Shapiro and Teixeira and the truncation-dependent Immirzi running of Benedetti and Speziale are Euclidean one-loop/RG calculations. The unresolved bridge is a derivation of a matched physical Lorentzian cosmological stress tensor and observable from those results [4–6]. We draw no dark-energy or birefringence inference from those running calculations.

IV. CLASSICAL SCALAR-SECTOR TRANSPARENCY

A. Statement

The precise local statement is as follows. Consider the classical ECH action with an invertible tetrad, canonical scalar matter, and a real finite nonzero constant γ, on a local patch with matched background, initial, and boundary data. Here standard boundary data means the usual falloff conditions with the boundary contribution to the first-order variation set to zero. Solve the algebraic connection equation before expanding in perturbations. Then the reduced action on the zero-spin branch is the Einstein–scalar action, so scalar equations and tensor evolution operators coincide with GR at every perturbative order for the same background and boundary data. Equality of right- and left-helicity *solutions*, rather than their evolution operators, additionally requires parity-symmetric initial data. The complex singular values $\gamma = \pm i$, nontrivial global or topological sectors, nonstandard boundary terms, quantum loops or anomalies, fermion sources, non-minimal matter, a dynamical Immirzi field, and propagating torsion are outside this statement.

This is equality after solving the Cartan equation—an on-connection-shell equality of local classical reduced actions—not an off-shell equality of the original first-order ECH and Einstein–scalar actions. Its all-orders content does not come from an order-by-order calculation: the Cartan constraint is algebraic, and the Bianchi identity used below is pointwise. The displayed linear tensor equation and second-order Holst expression are checks of those identities, not their foundation.

B. Proof (Scalar Sector)

1. **Zero spin density.** A canonical scalar field has zero spin density.

2. **Zero torsion.** The full Holst-modified connection equation uses the algebraic bivector operator $\mathcal{Q}_\gamma$ in Eqs. (2) and (3). For real finite nonzero γ its inverse exists because $1 + \gamma^2 > 0$. The zero scalar source therefore gives $e^{[I} \wedge T^{J]} = 0$, and the explicit dual-frame contraction below Eq. (3) proves that its invertible-tetrad kernel is trivial: $T^I = 0$. This holds at every classical field configuration in the stated local domain and hence before any perturbative expansion. The complex singular values $\gamma = \pm i$ are excluded.

3. **Connection reduces to Levi-Civita.** $\Gamma^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu}$.

4. **Holst term vanishes by the first Bianchi identity.** The Holst term evaluated with the Levi-Civita connection gives $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}(\mathring{\Gamma})$, which on a torsion-free connection *vanishes identically* by the first (algebraic) Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$: the cyclic-sum identity $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$ contracted with the totally antisymmetric $\epsilon^{\mu\nu\rho\sigma}$ leaves no non-trivial component. The algebraic Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$ holds for any torsionless connection, independently of metric compatibility; non-metricity does not invalidate the identity provided $T = 0$. The Holst dual contraction is therefore identically zero on the Levi-Civita connection (not merely a boundary term), so it contributes nothing to the action at any order. This Bianchi-vanishing is distinct from the Pontryagin density, a separate invariant containing two curvature factors; see the explicit verification below.

No total-derivative argument is a load-bearing step in this proof. The operative statement is the pointwise algebraic Bianchi identity in Step 4. The Nieh–Yan boundary-form decomposition quoted below is retained only to distinguish related densities; at nonzero torsion it does not replace the Cartan equation or make the full Holst density a boundary term.

C. Extension to Tensor Sector

The same four steps apply to tensor perturbations. As an illustrative source-free *linear* specialization on an FRW background, $T = 0$ gives the standard GR tensor equation

$$h_{ij}'' + 2\mathcal{H}h_{ij}' + k^2 h_{ij} = 0 \tag{11}$$

(primes denote derivatives with respect to conformal time η, and $\mathcal{H} \equiv a'/a$ is the conformal Hubble rate; Fourier modes $h_{ij}(\mathbf{k}, \eta)$ follow the standard $e^{i\mathbf{k}\cdot\mathbf{x}}$ convention for the transverse-traceless amplitude; in cosmic time, using $dt = a\,d\eta$, the equivalent form is $\ddot{h}_{ij} + 3H\dot{h}_{ij} + (k^2/a^2)h_{ij} = 0$) has no parity-dependent modifications. The left- and right-helicity evolution operators are identical,

$$\mathcal{E}_R = \mathcal{E}_L = \partial_\eta^2 + 2\mathcal{H}\partial_\eta + k^2. \tag{12}$$

Thus minimal ECH produces no helicity-dependent propagation; $v_R = v_L$ follows only when the initial data are also parity symmetric. It generates no GW birefringence, tensor chirality, or temperature–B-mode (TB) and E-mode–B-mode (EB) CMB cross-power spectra from parity-symmetric initial data. The all-order statement is not the displayed linear equation: it is equality of the classical action and equations of motion to their GR forms on the torsion-free branch, before any perturbative expansion.

D. Explicit Verification: The Holst Term in Perturbation Theory

Expanding to second order, the Holst dual evaluates on the Levi-Civita connection ($T = 0$) as:

$$\mathcal{R}_{\mathrm{H}}(\mathring{\Gamma}) \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}(\mathring{\Gamma}) \;=\; 0 \quad \text{(identically, by the first (algebraic) Bianchi identity).} \tag{13}$$

This is the Bianchi-vanishing of the Holst dual contraction on a torsion-free connection. The equivalent Nieh–Yan identity $d(e_I \wedge T^I) = T_I \wedge T^I - e_I \wedge e_J \wedge R^{IJ}$ also gives zero pointwise when $T^I = 0$; it is not a separate boundary assumption. This one-curvature Holst contraction must not be confused with the Pontryagin density, which contains two curvature factors and can be nonzero pointwise. On $T = 0$ the Holst contraction is identically zero by Bianchi, contributing nothing to the local classical reduced equations of motion at any perturbation order. In particular, the cubic action for ζ (which determines the bispectrum) receives zero contribution from the Holst term in the stated canonical-scalar domain. The resulting bispectrum equation is therefore the standard GR one for matched background, initial, and boundary data.

E. What Would Break the Transparency

The statement need not survive fermionic spin sources, torsion kinetic terms, non-minimal matter couplings, a dynamical Immirzi field, nonstandard boundary terms, nontrivial global/topological sectors, or quantum loops and anomalies. Those cases change either the Cartan source, the algebraic character of the connection equation, or the local classical action and are not analyzed here.

F. Implications

The perturbation-transparency result establishes a clean dichotomy:

- *Perturbation observables* (C_ℓ^{TT}, C_ℓ^{EE}, P_k, bispectrum): after solving the Cartan equation, their local classical equations equal the GR equations at every perturbative order under the conditions in Sec. IV. Observable solutions also require the same background, initial, and boundary data.

- *Parity*: the two tensor-helicity evolution operators coincide. Equal helicity solutions require parity-symmetric initial data; any parity signal must enter through a sector excluded from this theorem.

V. DISCUSSION AND LIMITATIONS

The two results have different logical status. Algebraic torsion elimination is an action-level calculation. The finite-density estimate then follows from dimensional scaling of that derived contact operator, whereas the NJL statement additionally assumes a specific mean-field reduction and regulator. The transparency result is exact within its classical domain because both load-bearing steps—zero scalar spin current and the algebraic Bianchi identity—hold pointwise. Its novelty is not a new curvature identity; its utility is to make the dimensional benchmark and claim boundary explicit for the constant-Immirzi, canonical-scalar branch.

Several stronger conclusions are not supported. This paper does not enumerate all diffeomorphism-invariant operators, establish vacuum stability beyond the stated NJL truncation, or constrain non-minimal fermion couplings. It does not analyze propagating torsion, a dynamical Immirzi field, quantum loops or anomalies in the scalar sector, or boundary/topological sectors with nontrivial global data. Most importantly, it does not derive a late-time vacuum energy, cosmic birefringence, or a bounce observable from ECH. These are residual research problems, not closures hidden by terminology.

VI. CONCLUSIONS

Minimal ECH gravity supplies a definite, Planck-suppressed axial contact interaction after algebraic torsion is integrated out. At the stated $n_\psi = 100\,\mathrm{cm}^{-3}$ illustrative normalization its coefficient-one scale is $3.6 \times 10^{-69}\rho_\Lambda$, scaling as n_ψ^2; this is not an equation-of-state or vacuum-stress calculation. Separately, its scalar projection has no nonzero real homogeneous solution in the declared direct-channel, hard-four-momentum- cutoff, standard mean-field gap equation because its coupling is repulsive. This conditional conclusion does not use a blanket magnitude-subcriticality claim: at the retained $\Lambda = M_{\rm Pl}$ bookkeeping ceiling the $N_f N_c = 9$ coefficient magnitude is supercritical, but no above-Planck cutoff is evaluated. Separately, canonical scalar matter sources no torsion, and the constant-Immirzi Holst term vanishes on the resulting torsion-free branch by the first Bianchi identity, leaving the local classical scalar equations and tensor evolution operators unchanged for the matched data and scope stated in Sec. IV.

These are the complete claims of this paper. The evaluated finite-density benchmark is far below the observed dark-energy scale, but no conclusion about a coherent mean field, vacuum stress tensor, or equation of state is drawn from $\langle J^5 \rangle \approx 0$. Independently, the classical Holst sector leaves scalar/tensor perturbations unchanged on the stated torsion-free scalar branch after the algebraic connection has been solved; this is not an off-shell identity of the original first-order actions. It does not yield a bounce observable there. This does not constitute an operator-complete no-go or an ECH explanation of dark energy. The unresolved step in proposed running-based extensions is the derivation of a matched Lorentzian cosmological stress tensor and observable from the underlying one-loop/RG calculation. Until that step exists, the honest conclusion is a specific contact-channel benchmark and a narrow classical transparency identity.

Data and Code Availability

The algebraic checks used here are committed at `fierz_lemma_check.py` and `njl_gap_equation_route1.py`; the latter writes the cutoff-ceiling and density record `njl_gap_equation_route1_results.json`. These exact files are frozen at immutable repository commit `7befce143848`. No companion computation is required for either action-level result in this Note.

Appendix A: Fierz Rearrangement of the Minimal Axial Contact Operator

On the 16-dimensional Clifford basis, grouped into the five Lorentz classes (S, V, T, A, P), we first state the c-number spinor rearrangement in the normalized convention of Nieves and Pal [7, 8]: $e_I(1234) = \sum_J (F_{\rm c})_{IJ} e_J(1432)$, with rows I the source classes and columns J the produced classes,

$$F_{\rm c} = \frac{1}{4}\begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 & 1 \\ 4 & -2 & 0 & -2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ -4 & -2 & 0 & -2 & 4 \\ 1 & -1 & \frac{1}{2} & 1 & 1 \end{pmatrix}, \qquad F_{\rm c}^2 = \mathbb{1}. \tag{A1}$$

In that normalized basis $\Gamma^\mu_A = i\gamma^\mu\gamma^5$ and $n_A = -i$, so the two factors combine to make e_A exactly the physical $(\bar\psi\gamma^\mu\gamma^5\psi)(\bar\psi\gamma_\mu\gamma^5\psi)$ operator; the phases are already absorbed in $F_{\rm c}$. Turning this c-number identity into an anticommuting field-operator identity requires one Grassmann exchange, so $F_{\rm op} = -F_{\rm c}$. The axial source is the fourth *row*, not the fourth column. In the resulting declared exchange-channel operator ordering,

$$(J^5 \cdot J^5) \xrightarrow{\text{Fierz}} SS + \tfrac{1}{2}VV + \tfrac{1}{2}AA - PP. \tag{A2}$$

The scalar bridge is therefore visible in three steps:

$$\begin{aligned} (F_{\rm c})_{AS} &= \tfrac{1}{4}(-4) = -1, \\ (F_{\rm op})_{AS} &= -(F_{\rm c})_{AS} = +1, \\ G_s &= \left(-\tfrac{3\kappa}{16}\right)(+1). \end{aligned} \tag{A3}$$

Here the factor 1/4 is the normalized gamma-trace factor in Eq. (A1), and the second minus sign is the single Grassmann exchange. Multiplying the resulting scalar coefficient +1 by the contact prefactor in Eq. (8) gives $\mathcal{L}_S = (-3\kappa/16)(\bar\psi\psi)^2 \equiv G_s(\bar\psi\psi)^2$, hence $G_s = -3\kappa/16$ in the gap-equation convention used below. The coefficients are dimensionless rationals, so rearrangement preserves the common $\kappa = 8\pi/M_{\rm Pl}^2$ prefactor. The matrix orientation, Grassmann sign, scalar gamma-trace projection, and channel coefficients are checked directly by `fierz_lemma_check.py`.

Equation (A2) is an operator-order-specific Dirac-algebra identity, not an assertion that one mean-field factorization is basis-independent. The gap-equation test below uses only its scalar exchange coefficient in the displayed direct-channel convention. We report the axial entry only through $|G_A|$ because its sign is convention-bound and no axial gap equation is solved. Nor does this identity enumerate derivative operators, curvature–current operators, independent flavor/color contractions, non-minimal torsion irreducible components, or the gravitational EFT.

Appendix B: Regulated Standard Mean-Field NJL Check

For the maximal Einstein–Cartan contact coefficient in Eq. (9), the scalar coefficient selected by Eq. (A2) is

$$G_{\rm scalar} = \left(-\frac{3}{16}\right)(+1)\kappa = -\frac{3}{16}\kappa. \tag{B1}$$

Consider the standard hard-Euclidean-four-momentum-cutoff NJL model with N_f flavors, N_c colors, and scalar contact term $G_s(\bar\psi\psi)^2$. In the chiral limit the mean-field gap equation is

$$M = G_s N_f N_c \frac{M}{2\pi^2}\left[\Lambda^2 - M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right)\right], \tag{B2}$$

which gives the bifurcation threshold

$$G_{\rm crit} = \frac{2\pi^2}{N_f N_c \Lambda^2} \tag{B3}$$

TABLE I. Coefficient-to-threshold diagnostics, with $R_S = |G_{\text{scalar}}|/G_{\text{crit}}$ and $R_A = |G_A|/G_{\text{crit}}$, where $|G_A| = 3\kappa/32$ and G_{crit} is the *scalar* threshold. Thus R_A is a coefficient-magnitude benchmark, not a derived axial-vector condensation threshold. Every row uses $\Lambda = M_{\text{Pl}}$ only as a bookkeeping ceiling; no controlled UV extension above that ceiling is implied. This table tests neither coherent axial order nor a cosmological stress tensor or equation of state.

$N_f N_c$	Λ/M_{Pl}	R_S	R_A
1	1	0.239	0.119
3	1	0.716	0.358
9	1	2.15	1.07

in this normalization [9–11]. Here $M = -2G_s\langle\bar{\psi}\psi\rangle$ and the hard four-dimensional Euclidean ball gives $\langle\bar{\psi}\psi\rangle = -(N_f N_c M/4\pi^2)[\Lambda^2 - M^2\ln(1+\Lambda^2/M^2)]$, fixing every factor in Eq. (B3). The scalar coefficient in Eq. (B1) is negative and therefore repulsive in this convention. Discarding that sign as a conservative magnitude test gives

$$\frac{|G_{\text{scalar}}|}{G_{\text{crit}}} = \frac{3N_f N_c}{4\pi}\frac{\Lambda^2}{M_{\text{Pl}}^2}, \tag{B4}$$

where the unreduced mass convention $\kappa = 8\pi/M_{\text{Pl}}^2$ has been used. The coupling in this diagnostic is deliberately the maximal Einstein–Cartan limit. We take $N_f N_c = 1, 3, 9$ at the bookkeeping ceiling $\Lambda = M_{\text{Pl}}$. No cutoff above M_{Pl} is evaluated because it would lie outside the controlled contact-EFT domain.

The scalar magnitude is therefore formally supercritical only in the $N_f N_c = 9$ cutoff-ceiling row, where the axial benchmark also exceeds unity. The old blanket magnitude-subcritical statement is false. The scalar-condensate conclusion instead follows from the separate sign result: inserting $G_{\text{scalar}} < 0$ into the real homogeneous scalar gap equation admits no nonzero solution. The pseudoscalar coefficient magnitude equals the scalar magnitude.

The result is reproducible with `njl_gap_equation_route1.py`. Its interpretation is strictly standard mean field with this regulator and channel convention. A different Fierz-complete truncation, exchange treatment, multi-species model, or nonperturbative completion can change the mean-field organization, so the calculation is not presented as a regulator- or basis-independent exclusion of every condensate mechanism. No alternate regulator is evaluated here, so we make no claim about how one would change the stability condition.

[1] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, General relativity with spin and torsion: Foundations and prospects, Reviews of Modern Physics **48**, 393 (1976).

[2] F. W. Hehl and B. K. Datta, Nonlinear spinor equation and asymmetric connection in general relativity, J. Math. Phys. **12**, 1334 (1971).

[3] L. Freidel, D. Minic, and T. Takeuchi, Quantum gravity, torsion, parity violation and all that, Physical Review D **72**, 104002 (2005), arXiv:hep-th/0507253 [hep-th].

[4] I. L. Shapiro and P. M. Teixeira, Quantum Einstein-Cartan theory with the Holst term, Classical and Quantum Gravity **31**, 185002 (2014), arXiv:1402.4854 [gr-qc].

[5] D. Benedetti and S. Speziale, Perturbative running of the Immirzi parameter, J. Phys. Conf. Ser. **360**, 012011 (2012), arXiv:1111.0884 [hep-th].

[6] D. Benedetti and S. Speziale, Perturbative quantum gravity with the Immirzi parameter, JHEP **2011** (6), 107, arXiv:1104.4028 [hep-th].

[7] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, 1980).

[8] J. F. Nieves and P. B. Pal, Generalized fierz identities, Am. J. Phys. **72**, 1100 (2004), arXiv:hep-ph/0306087 [hep-ph].

[9] S. P. Klevansky, The Nambu-Jona-Lasinio model of quantum chromodynamics, Rev. Mod. Phys. **64**, 649 (1992).

[10] T. Hatsuda and T. Kunihiro, QCD phenomenology based on a chiral effective Lagrangian, Phys. Rept. **247**, 221 (1994).

[11] M. Buballa, NJL model analysis of dense quark matter, Phys. Rept. **407**, 205 (2005).