

The separate universe computes a different variable: an exact criterion for $\delta N = \zeta$ in non-attractor phases

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The isotropic separate universe (δN) computes δN_c , the e-fold count of a fluid worldline labelled by its position on the initial slice; the standard δN formula identifies δN_c with Maldacena's ζ . We show this identification is an exact linear-order, super-Hubble change of variable, $\delta N_c = \zeta_{L,f}[1 - I/3 + O(k_L^2/a^2 H^2)]$, with $I = \zeta_{L,f}^{-1} \int (\epsilon/c_s^2) \dot{\zeta}_L dt$ the ζ -growth-weighted mean: $\delta N_c = \zeta$ at linear order iff $I \rightarrow 0$. At second order the worldline label matters: the initial-position label (the separate universe's own label) composes the in-in bispectrum back to the isotropic separate-universe value, -5 , for every constant ϵ ; the final-position label differs by a translation term with zero monopole. We present one nontrivial check (matter-dominated contraction, $I \rightarrow \epsilon$, an $O(1)$ identification error) plus three consistency limits – attractor slow roll and ekpyrosis ($I = 0$, identity map) and ultra-slow-roll ($I = O(\sqrt{\epsilon_s \epsilon_f})$, not computed here) – and place the criterion against the non-attractor consistency-relation and beyond- δN gradient-expansion literature. There is no discrepancy in the physics, only in which variable δN_c is compared to.

I. MOTIVATION

The squeezed limit of the primordial bispectrum is governed, in single-clock inflation, by Maldacena's consistency relation, and can equivalently be computed by the separate-universe (δN) formalism, which treats the long-wavelength mode as a local shift of the background evolution [1]. In a non-attractor phase, where $\dot{\zeta} \neq 0$ on super-Hubble scales, two independent difficulties arise. First, Namjoo, Firouzjahi & Sasaki showed that ultra-slow-roll (USR) inflation violates the consistency relation, $f_{\text{NL}} = 5/2$, and that δN built from $N(\phi)$ alone misses this: the growing mode is an independent initial datum, requiring $N(\phi, \pi)$ to be correctly captured [3]. This initial-data requirement was extended to $P(X)$ non-attractors by Chen, Firouzjahi, Namjoo & Sasaki, who found $f_{\text{NL}} = 5(1 + c_s^2)/(4c_s^2)$ and confirmed δN with momentum agrees [4]. Pajer, Schmidt & Zaldarriaga separately argued that in genuine attractors the squeezed bispectrum is an unobservable coordinate artifact of the long mode [5], a statement later sharpened with conformal Fermi coordinates by Dai, Pajer & Schmidt [6]. Cai *et al.* tracked f_{NL} through smooth and sharp USR transitions and found δN and in-in agree throughout [9]. Bravo, Mooij, Palma & Pradenas derived a generalized consistency relation for non-attractor backgrounds and showed the canonical single-field local f_{NL} is removable by a change of physical coordinates [10, 11], and Pasaglia, Hu & Motohashi confirmed $\delta N(\phi, \pi)/\text{in-in}$ agreement for PBH-motivated USR scenarios [12]. Separately, Artigas, Grain & Vennin, and Jackson *et al.*, identified a phase-space separate-universe breakdown localized to a finite window of scales at slow-roll-to-USR transitions, traced to a momentum/initial-data mismatch [13, 14].

Second, and logically independent, a companion theory-audit note on this program found that the isotropic separate universe applied to a matter-dominated cosmological contraction ($w = 0$, $\epsilon = 3/2$) returns $f_{\delta N} = -5$ (initial-position label) while the from-scratch in-in squeezed bispectrum of ζ is $-35/16 + (15/16)\mu^2$ [23], a monopole gap of $25/8$ – an $O(1)$ discrepancy, not a small correction. Cai, Xue, Brandenberger & Zhang's Eq. (37) is the correct squeezed shape function – equal to the sum of their Eqs. (27)–(32) and to Li, Quintin, Wang & Cai's Eq. (4.19) at $c_s = 1$ [8, 20] – but their quoted amplitudes Eqs. (38)–(41) and Fig. 5, including the isocles value they print as $-35/8$, are uniformly twice Eq. (37) in every configuration, so the correct isocles amplitude is $-35/16$; [23] locates this unprinted factor of 2 downstream of Eq. (37), in the amplitude-conversion step rather than in any vertex or commutator, and the from-scratch shape $(-35/16 + \frac{15}{16}\mu^2)$, monopole $-15/8$ then matches Cai *et al.*'s Eq. (37) monomial by monomial. Li *et al.*'s $-35/16$ (their Eq. 5.1 at $c_s = 1$) evaluates the same Eq. (4.19); their rows are an independent general- (ϵ, c_s) derivation that coincides with Cai *et al.*'s at $\epsilon = 3/2$, not a reuse of Cai's rows. Were the from-scratch shape uniformly twice as large instead, the monopole would be $-15/4$ and the gap to $f_{\delta N}^{\text{init}} = -5$ would be $-5/4$, a ratio of $4/3$ rather than $8/3$. The linear criterion $\lambda = 1 - \epsilon/3$ is independent of any bispectrum; only the second-order -5 composition uses the in-in input. A second companion note derived the exact second-order map between the two variables and showed the gap decomposes into a linear rescaling and an explicit second-order term, both proportional to ϵ [24]. Every prior test of $\delta N/\text{in-in}$ agreement in the literature above is an inflationary non-attractor, where $\epsilon \ll 1$ throughout the growth of the long mode. This note asks the question directly: is there a single criterion, applicable to both inflationary and contracting non-attractors, that predicts when the two constructions agree and by how much they

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differ when they do not.

II. THE THREADING IDENTITY AND THE MAP

Comoving gauge, ADM variables $N = 1 + \alpha$, $N_i = \partial_i \psi + \tilde{N}_i$. The separate universe evolves each patch as a homogeneous FRW region identified by its fluid element's position x_i on a flat, super-Hubble initial slice Σ_i ($k_L \ll a_i H_i$); its output is the e-fold count δN_c of the fluid (normal) congruence, labelled by x_i , from Σ_i to the comoving final slice Σ_f . Along a fluid worldline $\dot{x}^i = -N^i$ the fully nonlinear relation between the local expansion and the volume element gives the exact identity

$$\delta N_c(x_i) = \zeta(t_f, x_f) - \frac{1}{3} \int_{t_i}^{t_f} \partial_i N^i(t, x(t)) dt, \quad x_f = x(t_f), \quad (1)$$

with $\delta \rho_c / \rho \propto \partial_i N^i$ up to a gradient term; $\partial_i N^i$ is the coordinate divergence of the contravariant shift N^i , and $u^\mu \parallel n^\mu$ (fluid four-velocity along the ADM normal) presumes a single scalar degree of freedom, i.e. no vorticity. On super-Hubble scales, $\partial_i N^i = (\epsilon/c_s^2)\dot{\zeta} - \nabla^2 \zeta / (a^2 H)$ for the $P(X)$ class (a genuine fluid with non-adiabatic pressure adds a term) [1, 2]. Splitting the interval at the comoving slice through x_i and integrating,

$$\delta N_c = \zeta_{L,f} \left[1 - \frac{I}{3} + O\left(\frac{k_L^2}{a^2 H^2}\right) \right], \quad I \equiv \frac{1}{\zeta_{L,f}} \int_{t_i}^{t_f} \frac{\epsilon}{c_s^2} \dot{\zeta}_L dt, \quad (2)$$

exact on super-Hubble scales for any single-field history from a flat, super-Hubble initial slice; the dropped gradient term is $O(k_L^2/a_i H_i^2)$ or $O(k_L^2/a_f H_f^2)$, whichever end of the interval dominates. $I \rightarrow 0$ (identity map) on any constant mode, including the attractor and the dominant ekpyrotic mode; $1 - \lambda = I/3$ reduces to $\epsilon/3$ for constant ϵ (i.e. $I \rightarrow \epsilon$), so $\lambda \equiv 1 - I/3 = 1 - \epsilon/3$, and to $\lambda_{\text{USR}} = 1 + \epsilon_f/3 - \sqrt{\epsilon_s \epsilon_f}/3$ for USR with $\epsilon = \epsilon_s (a/a_s)^{-6}$ from a slow-roll-frozen $\zeta_{L,i} \neq 0$ at a_s , $\zeta_L \propto a^3$ thereafter. At second order, for constant ϵ and $c_s = 1$, assuming $\zeta_L(t_i) = 0$ (a pure growing mode, no constant piece on the initial slice), solving the exact ADM constraints to $O(\zeta_L \zeta_S)$ and integrating Eq. (1) gives $f_{\delta N} = f^{\text{in-in}}/\lambda + f_{\text{map}}$, where f_{map} depends on how the fluid worldline is labelled. Labelled by *initial* position x_i – the label the separate universe uses, since it never learns where the fluid element ends up –

$$f_{\text{map}}^{\text{init}}(\epsilon, \mu) = \frac{5\epsilon}{4(3-\epsilon)} [(\epsilon-2) - \epsilon\mu^2], \quad (3)$$

and $f_{\delta N}^{\text{init}} \equiv f^{\text{in-in}}/\lambda + f_{\text{map}}^{\text{init}} = -5$, isotropic, for every constant ϵ . Labelled instead by *final* position $x_f = x(t_f)$, the two labels differ by a pure translation $\xi^i = -\int N_L^i dt$ along the long mode, adding a shear-plus-dilation term with zero monopole:

$$f_{\text{map}}^{\text{fin}}(\epsilon, \mu) = -\frac{5\epsilon}{4}(1-\mu^2) = f_{\text{map}}^{\text{init}} - \frac{5\epsilon}{4(3-\epsilon)}(1-3\mu^2), \quad (4)$$

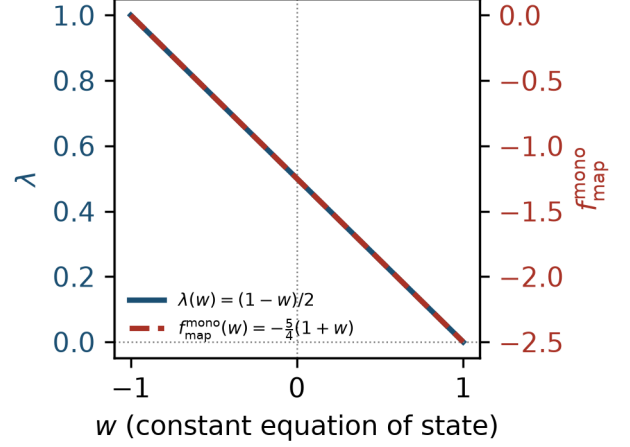


FIG. 1. The linear rescaling $\lambda(w) = (1-w)/2$ (dimensionless, left axis) and the label-independent second-order map monopole $f_{\text{map}}^{\text{mono}}(w) = -\frac{5}{4}(1+w)$ (dimensionless, right axis), Eq. (5), for constant equation of state $-1 \leq w \leq 1$ at $c_s = 1$. At $w = -1$ (attractor limit, $\epsilon = 0$) $f_{\text{map}}^{\text{mono}}$ vanishes and $\lambda \rightarrow 1$, so the map is the identity; λ vanishes at $w = 1$ (kination, the $m = 3/\epsilon - 1 = 0$ constant mode), a formal limit of the constant- ϵ formula rather than a physically realized loss of linear response.

giving $f_{\delta N}^{\text{fin}} = -\frac{15(\epsilon-4)}{4(\epsilon-3)} + \frac{15\epsilon}{4(3-\epsilon)}\mu^2$ ($-\frac{25}{4} + \frac{15}{4}\mu^2$ at $\epsilon = 3/2$); $\mu = \hat{k}_L \cdot \hat{k}_S$. Both labels give the same, label-independent monopole $-5\epsilon/6$; only the quadrupole is a labelling convention.¹ In terms of a constant equation of state w ($\epsilon = \frac{3}{2}(1+w)$, $c_s = 1$),

$$\lambda = \frac{1-w}{2}, \quad f_{\text{map}}^{\text{fin}} = -\frac{15}{8}(1+w)(1-\mu^2), \quad (5)$$

with label-independent monopole $f_{\text{map}}^{\text{mono}} = -\frac{5}{4}(1+w)$. Fig. 1 shows both across $-1 \leq w \leq 1$: at $w = -1$ ($\epsilon = 0$, de Sitter/attractor limit) $\lambda \rightarrow 1$ and $f_{\text{map}}^{\text{mono}} \rightarrow 0$, so the map is the identity; $\lambda \rightarrow 0$ at $w = 1$ (kination, $\epsilon = 3$, formally the $m = 3/\epsilon - 1 = 0$ constant mode, $\dot{\zeta}_L = 0$, $I = 0 - \lambda \rightarrow 0$ is a formal limit of the constant- ϵ formula rather than a physical loss of linear response), where the general- ϵ in-in monopole $-\frac{5}{18}(\epsilon-3)(\epsilon-6)$ itself vanishes.

III. THE CRITERION

Equations (2)–(4) give a single statement:

The map $\delta N_c = \zeta_{L,f} [1 - I/3 + O(k_L^2/a^2 H^2)]$, $I = \zeta_{L,f}^{-1} \int (\epsilon/c_s^2) \dot{\zeta}_L dt$, is exact and invertible for any single-field history ($\lambda \equiv 1 - I/3 \neq 0$); it is not a discrepancy in

¹ Each of the five geometric contributions to $f_{\text{map}}^{\text{fin}}$ carries an overall factor of ϵ : the lapse, the second-order metric perturbation, the gradient term, the final-position worldline displacement, and the initial-label translation [24].

the physics, only a statement about which variable δN_c is. The isotropic separate universe (δN , with $N(\phi, \pi)$) computes δN_c , and the standard δN formula identifies δN_c with ζ . That identification is exact iff $I \rightarrow 0$; it carries an $O(1)$ error in the identification – not in either variable separately – whenever $I = O(1)$, e.g. every constant- ϵ contraction with a growing ζ . When $I \ll 1$ (attractor and dominant ekpyrotic mode: $\dot{\zeta}_L = 0$, $I = 0$ exactly; USR: $\epsilon \ll 1$ throughout the growth) the identification holds to $O(I)$.

Via the Hamiltonian constraint on comoving slices, $I = O(1)$ is equivalent to the long mode’s Bardeen potential growing as $\Psi_L = O((aH/k)^2) \zeta_L$, the textbook signature of a non-attractor contraction [17, 18], and $I \rightarrow 0$ recovers the gradient-expansion assumption of Lyth, Malik & Sasaki [15], following Salopek & Bond [16]. The final slice used throughout is uniform- ϕ (comoving), which coincides with uniform- ρ only at $I \rightarrow 0$; at $I = O(1)$ the two differ already at $O(k^0)$, so Lyth–Malik–Sasaki’s $\delta N = \zeta_{ud}$ (uniform-density $\rightarrow$ uniform-density) is not the object compared here, and this is a second, independent $O(1)$ slice ambiguity of the same order as the effect this note isolates. Two logically distinct requirements are thus needed for the standard δN formula to give ζ in a non-attractor phase: (i) $N(\phi, \pi)$, to capture the growing mode as an independent initial datum [3]; (ii) $I \rightarrow 0$, so that $\delta N_c = \zeta$. USR satisfies (ii) approximately because $\epsilon \ll 1$ while ζ grows – not because the comoving shift is small, which it is not: the shift itself is $O(1/k_L)$ in every non-attractor, USR included, but its coefficient $\epsilon \dot{\zeta}_L$ is what controls I , and this coefficient is small in USR precisely when its growing-mode initial-data problem (i) is largest. Every inflationary test cited in Sec. I checked (i) with (ii) automatically small; none tested a regime with $I = O(1)$.

A distinct second-order question is the choice of final e-fold surface itself: continuing the fluid-congruence field N_c onto the uniform-density surface $\rho = \bar{\rho}(t_f)$ (rather than uniform- ϕ) to second order gives $f_{NL}^\rho = \frac{5}{24}(2\epsilon - 15)$, isotropic, i.e. $-\frac{5}{2}$ at $\epsilon = \frac{3}{2}$ (initial-position label) – an $O(1)$ slice dependence at second order as well as at first, additional to and independent of the linear-order criterion above (Appendix A 5).

Table I summarizes one nontrivial check plus three consistency limits by exact symplectic (script and manifest in the reproducibility statement). The $w = 0$ scalar case (a $w = 0$, $c_s = 1$ field, not true dust: genuine dust has $c_s \rightarrow 0$, where $I = \zeta^{-1} \int (\epsilon/c_s^2) \dot{\zeta} dt$ diverges and the criterion as stated does not apply) is the one already reported in Ref. [23]: the isotropic separate universe gives $f_{\delta N}^{\text{init}} = -5$ for every constant $\epsilon \in (1, 3)$, while the in-in monopole is $f_{\text{mono}}^{\text{in-in}} = -\frac{5}{18}(\epsilon - 3)(\epsilon - 6) = -15/8$ at $\epsilon = 3/2$, so the identification error, $-5 - (-15/8) = -25/8$, is a factor 8/3; the growing-mode monopole and the separate-universe value coincide only in the attractor limit $\epsilon \rightarrow 0$, where both are computed on the constant mode ($I = 0$) rather than by extrapolating this $\epsilon = 3/2$ result. USR agrees because $\epsilon \ll 1$ throughout

the growth of the long mode – its row tests the size of the $O(\epsilon)$ correction to the map, not a constraint from Namjoo-Firouzjahi-Sasaki’s leading-order $f_{NL} = 5/2$ itself [3]. Attractor slow roll is the identity map, consistent with Maldacena’s consistency relation [1]. Ekpyrotic contraction ($\epsilon \gg 3$) has $I = 0$ because the physical long mode sits on the constant- ζ solution, reproducing the attractor behavior of single-field ekpyrosis found by Creminelli, Nicolis & Zaldarriaga [19]; with $\dot{\zeta}_L = 0$ the row is a consistency check on the definition of I rather than a nontrivial test of the criterion. The non-dominant, growing- Ψ ekpyrotic mode would have $I = O(1)$ by the same criterion, but never sets $\zeta_L(t_f)$ and so is not physically realized. λ_{USR} assumes ζ_L is purely the growing a^3 mode from a slow-roll-frozen value at a_s ; a slow-roll-to-USR transition with a non-negligible decaying piece is not covered.

IV. WHAT IS NEW

Relative to Takamizu, Mukohyama, Sasaki & Tanaka and Naruko, Takamizu & Sasaki, who computed the $O(\epsilon_{\text{grad}}^2)$ (gradient-expansion, $m = 2$) correction beyond the leading-order δN /separate-universe approximation for the superhorizon non-Gaussianity [21, 22], the term this note restores, $-\nabla^2 \zeta / (a^2 H)$ in $\partial_i N^i$, is the same known next-to-leading-order gradient-expansion object; the point here is that in a non-attractor contraction the *other*, retained term, $(\epsilon/c_s^2) \dot{\zeta}$, is already $O(k^0)$ and $O(1)$ in ϵ , so the leading-order ($m = 0$) identification $\delta N_c = \zeta$ can fail at $O(1)$ without needing the $m = 2$ correction at all; the two effects are independent and this note isolates the first one with a single closed-form criterion I . Relative to Refs. [3, 4], this note identifies a second, independent source of identification error between δN_c and ζ in non-attractors – controlled by I rather than by initial data – and shows it is $O(\epsilon)$ in USR (hence invisible to their test) and $O(1)$ in a matter-like contraction. Relative to Ref. [5], the attractor case reduces to the vanishing of the non-attractor parameter $\Theta \equiv \epsilon \dot{\zeta}_L / (H \zeta_L)$, i.e. $\Theta = 0$ and the identity map, so the two curvature variables coincide there and the only remaining question is observability; away from the attractor they already differ at $O(1)$ before any observability argument, by an explicit, time-dependent, non-local map along the fluid worldline. Relative to Ref. [6], this note supplies the input a conformal-Fermi-coordinate treatment of a genuine non-attractor would need: the $O(k^0)$ divergence of the long mode’s comoving shift and the resulting displacement and dilation of the local frame. Relative to Dai, Pajer & Schmidt’s “On separate universes” [7], which states the conditions under which a long-wavelength mode is locally a separate FRW universe, the assumption that fails once $I = O(1)$ is theirs (b): the long mode is not adiabatic on the fluid worldline used here once $\epsilon/c_s^2 \dot{\zeta}_L$ is $O(k^0)$, so the local patch is still a separate universe but the map from its clock to the global ζ picks up the I -dependent rescal-

TABLE I. Validation of the criterion on four backgrounds (constant ϵ , $c_s = 1$ except USR). I is the ζ -growth-weighted mean of ϵ/c_s^2 ; $\lambda = 1 - I/3$; $f^{\text{in-in}}(\mu)$ the from-scratch in-in bispectrum and $f_{\text{mono}}^{\text{in-in}}$ its monopole; $f_{\delta N}^{\text{init}}$ the isotropic separate-universe value (initial-position label, the label the separate universe computes).

case	I	λ	$f^{\text{in-in}}(\mu)$	$f_{\text{mono}}^{\text{in-in}}$	$f_{\delta N}^{\text{init}}$	result
$w=0$ scalar, $c_s=1$, $\epsilon = 3/2$	3/2	1/2	$-35/16 + \frac{15}{16}\mu^2$	-15/8	-5	$I = O(1)$: identification off by 25/8
USR, $\epsilon \propto a^{-6}$ (not computed here)	$\ll 1$	$1 + O(\sqrt{\epsilon_s \epsilon_f})$	5/2	5/2	5/2	$I = O(\sqrt{\epsilon_s \epsilon_f})$: agree to that order
attractor slow roll	0	1	$\frac{5}{12}(1 - n_s)$	same	same	$I = 0$: identity map
ekpyrosis, dominant mode	0	1	n/a ($\dot{\zeta}_L = 0$)	same	same	$I = 0$: consistency check ($\dot{\zeta}_L = 0$ trivially)

ing and translation of Eqs. (2)–(4) rather than reducing to the identity. Relative to Refs. [9, 12], the agreement found there is explained as the $I \ll 1$ corner of the criterion rather than a coincidence of the USR transition. Relative to Refs. [10, 11], the coordinate-removal argument for the canonical single-field local f_{NL} is $O(\epsilon)$ -controlled in inflation; whether an analogous physical-coordinate argument removes the $O(1)$ in-in monopole in a matter-like contraction is a distinct, open question, not addressed here. Relative to Refs. [13, 14], the mechanism and order differ: their breakdown is localized to a finite window of scales at a slow-roll-to-USR transition and traced to a momentum mismatch, while the criterion here applies to all super-Hubble scales at $O(k^0)$, at constant ϵ , with no transition required.

V. LIMITS

The second-order map, Eqs. (3)–(5), is exact for constant ϵ and $c_s = 1$ (the scalar-field realization of a given w); it is not the second-order map for a genuine fluid with $c_s^2 = w \neq 1$, which needs the c_s -dependent second-order ADM constraints – an open, bounded extension. The USR second-order statement in Sec. III is structural (every frozen second-order kernel is proportional to ϵ , and vanishes as $\epsilon \rightarrow 0$), not a re-solve with a time-dependent $\epsilon(t) = \epsilon_s(a/a_s)^{-6}$; a full time-dependent- ϵ second-order calculation was not attempted. The linear criterion, Eq. (2), holds on super-Hubble scales for any single-field history and any c_s , from a flat, super-Hubble initial slice, up to the dropped $O(k_L^2/a^2 H^2)$ gradient term bounded in Sec. II; it is not literally exact at all k_L .

REPRODUCIBILITY STATEMENT

The linear criterion, second-order map, general- w closed forms, and all four validations are computed by exact sympy in the committed script [research/theory_audit/separate_universe_](#)

[failure_criterion_2026_09_04.py](#) (SHA-256 21668ab6...), with results in the companion [.json](#) (SHA-256 4e2a49be...) and derivation in the companion note of the same base name ([.md](#)); the second-order threading map is derived in [research/theory_audit/threading_map_second_order_2026_09_04.{md,py,json}](#), and the in-in inputs in [research/theory_audit/fnl_monopole_adjudication_2026_09_03.md](#). The label-resolved second-order composition (both worldline labels, Eqs. (3)–(4)) and the restored gradient term (Eq. (2)) are derived and sympy-validated in [research/theory_audit/psu_gates_S1_S2_2026_09_04.{py,json}](#) (SHA-256 4f641022... / f4164019...). Figure 1 is generated by [make_fig_lambda_fmap.py](#) in this directory, transcribing the general- w closed forms with no new derivation. Manifest: [reproducibility/manifests/experiments/lift2-separate-universe-failure-criterion.json](#) and [reproducibility/manifests/experiments/psu-gates-s1-s2-label-composition-criterion.json](#). Local CPU, \$0, under 5 seconds total.

AI USAGE DISCLOSURE

The scripts verify algebra: the symbolic manipulations underlying the linear criterion, the second-order map for both worldline labels, and the four validations were performed by deterministic sympy scripts written and executed under the author’s direction, with every intermediate result printed and cross-checked against the frozen inputs. The physical identification of the separate universe’s output with the initial-position-label δN_c (Sec. II), and the framing of the criterion as a change of variable rather than a failure (Sec. III), are the author’s arguments, not sympy output. Claude (Anthropic) assisted with literature cross-checks, prose drafting, and LaTeX formatting under the author’s direction; all physics content, equations, and numerical results were verified by the author against the committed script output.

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Appendix A: Self-contained second-order derivation

This appendix transcribes, from the committed companion notes cited above, the material a standalone reader needs to verify Eq. (3)–(4) and the -5 composition without the external GitHub notes.

1. The second-order lapse/shift solution and the exact worldline identity

Comoving gauge ($\delta\phi = 0$), Maldacena variables $h_{ij} = a^2 e^{2\zeta} \delta_{ij}$, $N = 1 + \alpha$, $N_i = \partial_i \psi + \tilde{N}_i$; at first order $\alpha = \dot{\zeta}/H$, $\psi = -\zeta/H + \chi$, $\partial^2 \chi = a^2 \epsilon \dot{\zeta}$. The zero-shift threading is the fluid (normal) congruence, $n^\mu = (1/N, -N^i/N)$, $dx^i/dt = -N^i$. The expansion of the normal congruence is $K = \nabla_\mu n^\mu = \frac{1}{N\sqrt{h}}[\partial_t \sqrt{h} - \partial_i(\sqrt{h} N^i)]$; along a fluid worldline this gives the exact (fully nonlinear) identity

$$NK = \frac{d}{dt} \ln \sqrt{h} \Big|_{\text{worldline}} - \partial_i N^i. \quad (\text{A1})$$

With $\delta N_c = \int (K/3) N dt - \tilde{N}$ and $\ln \sqrt{h} = 3 \ln a + 3\zeta$, this is Eq. (1) of the main text. Every difference between ζ_{Mald} and δN_c is the divergence of the shift integrated along the fluid worldline: (i) $\partial_i N^i$ itself at second order, from the second-order lapse and shift solving the exact ADM Hamiltonian and momentum constraints; (ii) its evaluation along the worldline displaced by the long mode; (iii) the choice of label (final position x_f vs. initial position $x_i = x_f + \xi$), a rigid translation of the final field.

2. The five second-order kernel contributions (script: “threading map”)

Writing $\delta N_c^{(2)} = \mathcal{M} \zeta_L \zeta_S$, the kernel solving the exact ADM constraints to $O(\zeta_L \zeta_S)$ has five pieces of distinct geometric origin, each carrying an explicit overall factor of ϵ :

zlap the $-2\zeta \partial^2 \psi$ term from $e^{-2\zeta}$ in $N^i = h^{ij} N_j$ (local kernel $2\epsilon/3$); f_{NL} contribution $\frac{5\epsilon}{(3-\epsilon)^2}$, monopole the same (isotropic).

psi2 the second-order scalar shift $\partial^2 \psi_2/a^2$ from the constraints (non-local, $O(k_S/k_L)$ poles); contribution $\frac{5\epsilon(-2\epsilon^2+11\epsilon-24)}{8(3-\epsilon)^2} + \frac{5\epsilon(2\epsilon^2-9\epsilon+12)}{8(3-\epsilon)^2} \mu^2$, monopole $\frac{5\epsilon(-\epsilon^2+6\epsilon-15)}{6(3-\epsilon)^2}$.

grad $-2\partial_i \zeta \partial_i \psi_1/a^2$; contribution $-\frac{5\epsilon}{4(3-\epsilon)^2} + \frac{15\epsilon}{4(3-\epsilon)^2} \mu^2$, monopole 0.

wl_fin $\partial_i N_S^i$ read along the worldline displaced by $\Delta_L(t) = \int_t^{t_f} N_L dt'$; contribution $\frac{5\epsilon^2}{8(3-\epsilon)^2} - \frac{15\epsilon^2}{8(3-\epsilon)^2} \mu^2$, monopole 0.

lab_init (+ **wl_initextra**) the rigid translation $x_f \rightarrow x_i$ (initial-position label), $\propto (1 - 3\mu^2)$, monopole 0.

Totals. Final-position label: $f_{\text{map}} = -\frac{5\epsilon}{4} + \frac{5\epsilon}{4} \mu^2$, monopole $-5\epsilon/6$. Initial-position label: monopole again $-5\epsilon/6$; only the quadrupole changes, by the translation term. Every $1/k_L$ pole cancels between the two short legs.

3. The two worldline labels and the translation

A worldline can be labelled by its final position $x_f = x(t_f)$ (Eulerian, at the final comoving slice) or its initial position $x_i = x(t_i) = x_f + \xi$, with $\xi^i \equiv \int_{t_i}^{t_f} N_L^i dt$ (Lagrangian, the position on the initial slice where the separate-universe patches are laid out); for the long mode $N_L^i = a^{-2} \partial_i \chi_L$, $\partial^2 \chi_L = a^2 \epsilon \dot{\chi}_L$, giving $\xi(k_L) = -i \hat{k}_L \epsilon \zeta_L(k_L)/k_L$ at constant ϵ . The two labelled fields are related by a translation by this long-mode field; in the squeezed bispectrum the $O(k_S/k_L)$ poles cancel between the two short legs, the trace part vanishes at $n_s = 1$, and the shear gives the quadrupole-only translation term

$$T(\epsilon, \mu) \equiv f_{\text{map}}^{\text{init}} - f_{\text{map}}^{\text{fin}} = \frac{5\epsilon}{4(3-\epsilon)}(1-3\mu^2), \quad (\text{A2})$$

monopole 0 (all ϵ).

With $\lambda = 1 - \epsilon/3$ and $f^{\text{in-in}}(\mu, \epsilon) = \frac{5}{12}(\epsilon^2 \mu^2 - \epsilon^2 + 6\epsilon - 12)$, the per-label maps are

$$f_{\text{map}}^{\text{fin}}(\epsilon, \mu) = -\frac{5\epsilon}{4}(1-\mu^2), \quad (\text{A3})$$

$$f_{\text{map}}^{\text{init}}(\epsilon, \mu) = \frac{5\epsilon}{4(3-\epsilon)}[(\epsilon-2) - \epsilon\mu^2], \quad (\text{A4})$$

both with monopole $-5\epsilon/6$, and $f_{\delta N}^{\text{init}} = f^{\text{in-in}}/\lambda + f_{\text{map}}^{\text{init}} \equiv -5$ isotropically for every constant ϵ . The separate universe evolves each patch as a homogeneous FRW region identified by its fluid element's coordinates on the initial slice; it has no shift and never learns where that element ends up, so its output is δN_c labelled by *initial* position – the label for which the composition is exactly -5 .

4. The general- ϵ in-in shape and the identity-vs-fit check

The from-scratch in-in squeezed-limit shape used above, for general constant ϵ [23], is

$$f^{\text{in-in}}(\mu, \epsilon) = \frac{5}{12}(\epsilon^2 \mu^2 - \epsilon^2 + 6\epsilon - 12), \quad (\text{A5})$$

whose isotropy under the composition of Sec. II depends on this full shape, not only on its monopole $-\frac{5}{18}(\epsilon-3)(\epsilon-6)$ (the note this appendix draws on prints the monopole alone). The second-order map derivation (Secs. A1-A3 above) is computed from the exact ADM constraints alone and is validated symbolically before any in-in number is substituted: the constraint solver and the bispectrum assembly are written and frozen, then $f^{\text{in-in}}$ is read in only at the closure step. The map derivation therefore takes no in-in and no δN input; the composed

monopole -5 is a consequence of the map applied to the independently-derived in-in shape, not a fit to the separate universe's own -5 .

5. The general admixture: constant-mode kernel and the corrected normalisation

For a long mode with a constant admixture, $\zeta_L = C + G$ with $g \equiv G_f/\zeta_{L,f}$ the growing-mode fraction and $\lambda_g = 1 - \epsilon g/3$, the second-order shift kernel is bilinear, $M(g) = \frac{2}{3}\epsilon + gK_{\text{rest}}^{\text{grow}} + (1-g)K_c$, where K_c is the constant-mode piece, obtained by re-solving the ADM constraints at $(m_L, m_S) = (0, 3/\epsilon - 1)$; in the squeezed limit $K_c \rightarrow -\frac{2}{3}\epsilon$, independent of the short-mode history. This exactly cancels the constant-mode **zlap** contribution $2\epsilon/3$: a constant long mode carries no $O(k^0)$ threading map, as it must, since a constant ζ_L is a pure coordinate rescaling. The bispectrum normalisation for a mixed long mode is two-lambda, not $1/\lambda_g^2$: the long leg carries λ_g , the short (growing) leg $\lambda_1 \equiv 1 - \epsilon/3$,

$$f_{\text{map}}(g) = \frac{5}{12 \lambda_g \lambda_1} \frac{M(g; k_L, q)P(q) + M(g; k_L, p)P(p)}{P_S} \Big|_{\text{sq}} = \frac{g\lambda_1}{\lambda_g} f_{\text{map}}(1), \quad (\text{A6})$$

so $g = 1$ reproduces the -5 (initial label) / $-\frac{25}{4} + \frac{15}{4}\mu^2$ (final label) compositions above exactly, and $g = 0$ gives the null map $f_{\text{map}}(0) = 0$ for both labels – a purely constant long mode carries no super-Hubble signal (`psu_gates_S9_S10_2026_09_05.py`).

The uniform-density (-)slice question of Sec. II is distinct from this admixture: continuing N_c to second order on $\rho = \bar{\rho}(t_f)$ has squeezed monopole $f_{\text{NL}}^\rho = \frac{5}{8}(\epsilon - 7) = -\frac{55}{16}$ at $\epsilon = \frac{3}{2}$ (isotropic, initial label), reconciled by an independent adjudication (`research/theory_audit/a2_lapse_monopole_adjudication_2026_09_07.md`): the second-order threading map's long $\times$ short lapse monopole coefficient $A_2 = \epsilon(3-\epsilon)^2/3$ is the correct constraint-solve value (not $2(3-\epsilon)^2$), and the earlier initial-label figure $\frac{5}{24}(2\epsilon-15) = -\frac{5}{2}$ came from composing the map with the linear-mode weight $\lambda' = 2\lambda$ where the ρ -surface time shift, acting on the second-order curvature perturbation, instead carries weight 3λ ; correcting the weight closes the gap $5(6-\epsilon)/24$ exactly. An independent exact separate-universe solution reproduces both $-\frac{55}{16}$ (uniform-density slice) and -5 (comoving slice) for all ϵ from the same ADM monopole equations, so the two slice values, together with the in-in monopole $-\frac{15}{8}$ at dust, are three well-defined variables related by exact threading maps, not competing claims about one number.